

Small perturbation of a disordered harmonic chain by a noise and an anharmonic potential

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Abstract

We study the thermal properties of a pinned disordered harmonic chain weakly perturbed by a noise and an anharmonic potential. The noise is controlled by a parameter $\lambda \rightarrow 0$, and the anharmonicity by a parameter $\lambda' \leq \lambda$. Let κ be the conductivity of the chain, defined through the Green-Kubo formula. Under suitable hypotheses, we show that $\kappa = \mathcal{O}(\lambda)$ and, in the absence of anharmonic potential, that $\kappa \sim \lambda$. This is in sharp contrast with the ordered chain for which $\kappa \sim 1/\lambda$, and so shows the persistance of localization effects for a non-integrable dynamics.

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1 Introduction

The mathematically rigorous derivation of macroscopic thermal properties of solids, starting from their microscopic description, is a serious challenge [8, 17]. On the one hand, numerous experiments and numerical simulations show that, for a wide variety of materials, the heat flux J is related to the gradient of temperature T through a simple relation known as Fourier's law:

$$J = -\kappa(T) \nabla T,$$

where $\kappa(T)$ is the thermal conductivity of the solid. On the other hand, the mathematical understanding of this phenomenological law from the point of view of statistical mechanics is still lacking.

A one-dimensional solid can be modeled by a chain of oscillators, each of them being possibly pinned by an external potential, and interacting through a nearest neighbors coupling. The case of homogeneous harmonic interactions can be readily analyzed, but it has been realized that this very idealized solid behaves like a perfect conductor, and so violates Fourier's law [21]. To take account of physical observations, one thus needs to consider more elaborate models, where ballistic transport of energy is broken. Here are two possible directions.

On the one hand, adding some anharmonic interactions can drastically affect the conductivity of the chain [2, 19]. Unfortunately, the rigorous study of anharmonic chains is in general out of reach, and even numerical simulations do not lead to completely unambiguous conclusions. In order to draw some clear picture, anharmonic interactions are mimicked in [3, 6] by a stochastic noise that preserves total energy and possibly total momentum. The thermal behavior of anharmonic solids is, at a qualitative level, correctly reproduced by this partially stochastic model. By instance, the conductivity of the one-dimensional chain is shown to be positive and finite if the chain is pinned, and to diverge if momentum is conserved.

On the other hand, impurities constitute another element that can affect the conductivity of an harmonic chain. In [22] and [10], an impure solid is modeled by a disordered harmonic chain, where the masses of the atoms are random. In these models, localization of eigenmodes induce a dramatic fall off of the conductivity. In the presence of everywhere onsite pinning, it is known that the chain behaves like a perfect insulator (see Remark 1 after Theorem 1). The case of unpinned chain is more delicate, and turns out to depend on the boundary conditions [14]. The principal cases have been rigorously analyzed in [24] and [1].

The thermal conductivity of an harmonic chain perturbed by both disorder and anharmonic interactions is a topic of both practical and mathematical interest. We will in the sequel only consider a one-dimensional disordered chain with everywhere onsite pinning. Doing so we avoid the pathological behavior of unpinned one-dimensional chains, and we focus on a case where the distinction between ordered and disordered harmonic chain is the sharpest. We will consider the joint action of a noise

and an anharmonic potential ; we call λ the parameter controlling the noise, and λ' the parameter controlling the anharmonicity (see Subsection 2.1 below).

The disordered harmonic chain is an integrable system where localization of the eigenmodes can be studied rigorously [16]. However, if some anharmonic potential is added, very few appears to be known about the persistence of localization effects. In [13], it is shown through numerical simulations that an even small amount of anharmonicity leads to a normal conductivity, destroying thus the localization of energy. In [20], an analogous situation is studied and similar conclusions are reached. This is confirmed rigorously in [5], if the anharmonic interactions are replaced by a stochastic noise preserving energy. Nothing however is said there about the conductivity as $\lambda \rightarrow 0$. Later, this partially stochastic system has been studied in [12], where numerical simulations indicate that $\kappa \sim \lambda$ as $\lambda \rightarrow 0$.

Let us mention that, although the literature on the destruction of localized states seems relatively sparse in the context of thermal transport, much more is to find in that of Anderson's localization and disordered quantum systems (see [4] and references in [4, 12]). There as well however, few analytical results seem to be available. Moreover, the interpretation of results from these fields to the thermal conductivity of solids is delicate, in part because many studies deal with systems at zero temperature: the time evolution of an initially localized wave packet.

The main goal of this article is to establish that disorder strongly influences the thermal conductivity of a harmonic chain, when both a small noise and small anharmonic interactions are added. We will always assume that $\lambda' \leq \lambda$, meaning that the noise is the dominant perturbative effect. Our main results, stated in Theorems 1 and 2 below, are that $\kappa = \mathcal{O}(\lambda)$ as $\lambda \rightarrow 0$, and that $\kappa \sim \lambda$ if $\lambda' = 0$. Stricly speaking, our results do not imply anything about the case where $\lambda' > 0$ and $\lambda = 0$. However, in the regime we are dealing with, the noise is expected to produce interactions between localized modes, and so to increase the conductivity. We thus conjecture that $\kappa = \mathcal{O}(\lambda')$ in this later case. This is in agreement with numerical results in [20], where it is suggested that κ could even decay as $e^{-c/\lambda'}$ for some $c > 0$.

In the next chapter, we define the model studied in this paper, we state our results and we give some heuristic indications. The rest of the paper is then devoted to the proof of Theorems 1 and 2, in a way described after the statment of Theorem 2.

2 Model and results

2.1 Model

We consider a one-dimensional chain of N oscillators, so that a state of the system is characterized by a point

$$x = (q, p) = (q_1, \dots, q_N, p_1, \dots, p_N) \in \mathbb{R}^{2N},$$

where q_k represents the position of particle k , and p_k its momentum. The dynamics is made of a hamiltonian part perturbed by a stochastic noise.

The Hamiltonian. The Hamiltonian writes

$$\begin{aligned} H(q, p) &= H_{har}(q, p) + \lambda' H_{anh}(q, p) \\ &= \frac{1}{2} \sum_{k=1}^N \left(p_k^2 + \nu_k q_k^2 + (q_{k+1} - q_k)^2 \right) + \lambda' \sum_{k=1}^N \left(U(q_k) + V(q_{k+1} - q_k) \right), \end{aligned}$$

with the following definitions.

- The pinning parameters ν_k are i.i.d. random variables, which law is independent of N . One assumes this law to admit a bounded density, and that there exist constants $0 < \nu_- < \nu_+ < \infty$ such that

$$\mathbf{P}(\nu_- \leq \nu_k \leq \nu_+) = 1.$$

- The value of q_{N+1} depends on the boundary conditions (BC). For fixed BC, one puts $q_{N+1} = 0$. For periodic BC, one puts $q_{N+1} = q_1$. For further use, one also defines $q_0 = q_1$ for fixed BC, and $q_0 = q_N$ for periodic BC.
- One assumes that $\lambda' \geq 0$. The potentials U and V are symmetric, meaning that $U(-x) = U(x)$ and $V(-x) = V(x)$ for every $x \in \mathbb{R}$. One assumes that U and V belong to $\mathcal{C}_{temp}^\infty(\mathbb{R})$, the space of infinitely differentiable functions with polynomial growth. One assumes moreover that

$$\int_{\mathbb{R}} e^{-U(x)} dx < +\infty \quad \text{and} \quad \partial_x^2 U(x) \geq 0,$$

and that there exists $c > 0$ such that

$$c \leq 1 + \lambda' \partial_x^2 V(x) \leq c^{-1}.$$

For $x, y \in \mathbb{R}^d$, let $\langle x, y \rangle = x_1 y_1 + \dots + x_d y_d$ be the canonical scalar product of x and y . The harmonic hamiltonian H_{har} can also be written as

$$H_{har}(q, p) = \frac{1}{2} \langle p, p \rangle + \frac{1}{2} \langle q, \Phi q \rangle,$$

if one introduces the symmetric matrix $\Phi \in \mathbb{R}^{N \times N}$ of the form $\Phi = -\Delta + W$, where Δ is the discrete Laplacian, and W a random “potential”. The precise definition of Φ depends on the BC:

$$\begin{aligned} \Phi_{j,k} &= (2 + \nu_k) \delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} && \text{(fixed BC),} \\ \Phi_{j,k} &= (2 + \nu_k) \delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} - \delta_{1,N} - \delta_{N,1} && \text{(periodic BC),} \end{aligned}$$

for $1 \leq j, k \leq N$.

The dynamics. The generator of the hamiltonian part of the dynamics is written as

$$A = A_{har} + \lambda' A_{anh}$$

with

$$A_{har} = \sum_{k=1}^N (\partial_{p_k} H_{har} \cdot \partial_{q_k} - \partial_{q_k} H_{har} \cdot \partial_{p_k}) = \langle p, \nabla_q \rangle - \langle \Phi q, \nabla_p \rangle$$

and

$$A_{anh} = - \sum_{k=1}^N \partial_{q_k} H_{anh} \cdot \partial_{p_k} = - (\partial_x U(q_k) + \partial_x V(q_k - q_{k-1}) - \partial_x V(q_{k+1} - q_k)) \cdot \partial_{p_k}.$$

The generator of the noise is defined to be

$$\lambda Su = \lambda \sum_{k=1}^N (u(\dots, -p_k, \dots) - u(\dots, p_k, \dots)),$$

with $\lambda \geq \lambda'$. The generator of the full dynamics is given by

$$L = A + \lambda S.$$

We denote by $X_{(\lambda, \lambda')}^t(x)$, or simply by $X^t(x)$, the value of the Markov process generated by L at time $t \geq 0$, starting from $x = (q, p) \in \mathbb{R}^{2N}$.

Expectations. Three different expectations will be considered. We define

- μ_T : the expectation with respect to the Gibbs measure at temperature T ,
- \mathbb{E} : the expectation with respect to the realizations of the noise,
- \mathbb{E}_ν : the expectation with respect to the realizations of the pinnings.

We will also write \mathbb{E}_T for $\mathbb{E} \mu_T$.

The Gibbs measure μ_T is explicitly given by

$$\mu_T(u) = \frac{1}{Z_T} \int_{\mathbb{R}^{2N}} u(x) e^{-H(x)/T} dx,$$

where Z_T is a normalizing factor such that μ_T is a probability measure on \mathbb{R}^{2N} . We will need some properties of this measure. Let us write

$$Z_T^{-1} e^{-H(x)/T} = \rho'(p_1) \dots \rho'(p_N) \cdot \rho''(q),$$

with $\rho'(p_k) = e^{-p_k^2/2T} / \sqrt{2\pi T}$ for $1 \leq k \leq N$.

When $\lambda' = 0$, the density ρ' is Gaussian:

$$\rho'(q) = (2\pi T)^{-N/2} \cdot (\det \Phi)^{1/2} \cdot e^{-\langle q, \Phi q \rangle / 2T}.$$

Since $\nu_k \geq \nu_- > 0$, it follows from Lemma 1.1 in [9] that $|(\Phi^{-1})_{i,j}| \leq C e^{-c|j-i|}$, for some constants $C < +\infty$ and $c > 0$ independent of N . This implies in particular the decay of correlations

$$\mu_T(q_i q_j) = T(\Phi^{-1})_{i,j} \leq C T e^{-c|j-i|}.$$

When $\lambda' > 0$, the density ρ' is not Gaussian anymore. We here impose the extra assumption that ν_- is large enough. In that case, our hypotheses ensure that the conclusions of Theorem 3.1 in [7] hold: there exist constants $C < +\infty$ and $c > 0$ such that, for every $f, g \in \mathcal{C}_{temp}^\infty(\mathbb{R}^N)$ satisfying $\mu_T(f) = \mu_T(g) = 0$, one has

$$|\mu_T(f \cdot g)| \leq C e^{-c d(S(f), S(g))} \left(\mu_T(\langle \nabla_q f, \nabla_q f \rangle) \cdot \mu_T(\langle \nabla_q g, \nabla_q g \rangle) \right)^{1/2}. \quad (2.1)$$

Here, $S(u)$ is the support of a function u , defined as the smallest set of integers such that u can be written as a function of the variables x_l for $l \in S(u)$, whereas $d(S(f), S(g))$ is the smallest distance between any integer in $S(f)$ and any integer in $S(g)$. Using that $\mu_T(q_k) = 0$ for $1 \leq k \leq N$, one checks from (2.1) that every function $u \in \mathcal{C}_{temp}^\infty(\mathbb{R}^N)$ with given support independent of N is such that $\|u\|_{L^1(\mu_T)}$ is bounded uniformly in N .

The current. The local energy e_k of atom k is defined as

$$e_k = e_{k,har} + \lambda' e_{k,anh}$$

with

$$e_{k,har} = \frac{p_k^2}{2} + \nu_k \frac{q_k^2}{2} + \frac{1}{4}(q_k - q_{k-1})^2 + \frac{1}{4}(q_{k+1} - q_k)^2 \quad \text{for } 2 \leq k \leq N-1,$$

and

$$e_{k,anh} = U(q_k) + \frac{V(q_k - q_{k-1})}{2} + \frac{V(q_{k+1} - q_k)}{2} \quad \text{for } 2 \leq k \leq N-1.$$

For periodic B.C., these expressions are still valid when $k = 1$ or $k = N$. For fixed B.C. instead, one multiplies by a factor 2 all the terms involving the differences $(q_0 - q_1)$ or $(q_{N+1} - q_N)$ in the previous expressions. These definitions ensure that the total energy is the sum of the local energies.

The definition of the dynamics implies that

$$de_k = (j_{k-1} - j_k) dt$$

for local currents

$$j_k = j_{k,har} + \lambda' j_{k,anh}$$

defined as follows for $0 \leq k \leq N$. First, for $1 \leq k \leq N-1$,

$$j_{k,har} = \frac{1}{2}(p_k + p_{k+1})(q_k - q_{k+1}) \quad \text{and} \quad j_{k,anh} = \frac{1}{2}(p_k + p_{k+1}) \partial_x V(q_k - q_{k+1}). \quad (2.2)$$

Next, $j_{0,1} = j_{N,N+1} = 0$ for fixed B.C. Finally, j_0 and j_N are still given by (2.2) for periodic B.C., with the conventions $p_0 = p_N$ and $p_{N+1} = p_1$. The total current and the rescaled total current are then defined by

$$J_N = J_{N,har} + \lambda' J_{N,anh} = \sum_{k=1}^N j_{k,har} + \lambda' \sum_{k=1}^N j_{k,anh} \quad (2.3)$$

$$\mathcal{J}_N = \mathcal{J}_{N,har} + \lambda' \mathcal{J}_{N,anh} = \frac{J_{N,har}}{\sqrt{N}} + \lambda' \frac{J_{N,anh}}{\sqrt{N}}. \quad (2.4)$$

2.2 Results

For a given realization of the pinnings, the (Green-Kubo) conductivity $\kappa = \kappa(\lambda, \lambda')$ of the chain is defined as

$$\kappa(\lambda, \lambda') = \frac{1}{T^2} \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \kappa_{t,N}(\lambda, \lambda') = \frac{1}{T^2} \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X_{(\lambda, \lambda')}^s \, ds \right)^2 \quad (2.5)$$

if this limit exists. One expects the choice of the boundary conditions to play no role in this formula since the volume N is sent to infinity for fixed time. The homogenized conductivity is defined by replacing \mathbb{E}_T by $\mathbb{E}_\nu \mathbb{E}_T$ in (2.5). By ergodicity, one expects that the conductivity and the homogenized conductivity coincide for almost all realization of the pinnings (see [5]). The dependence of $\kappa(\lambda, \lambda')$ on the temperature T will not be analyzed in this work, so that one can consider T as a fixed given parameter.

We first obtain an upper bound on the homogenized conductivity.

Theorem 1. *Let $0 \leq \lambda' \leq \lambda$. With the assumptions introduced up to here, if ν_- is large enough, and for fixed boundary conditions,*

$$\frac{1}{T^2} \limsup_{t \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X_{(\lambda, \lambda')}^s \, ds \right)^2 = \mathcal{O}(\lambda) \quad \text{as} \quad \lambda \rightarrow 0. \quad (2.6)$$

Remarks. 1. When $\lambda = 0$, the proof (see Section 3) actually shows that

$$\frac{1}{T^2} \limsup_{N \rightarrow \infty} \mathbb{E}_\nu \mu_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X_{(0,0)}^s \, ds \right)^2 = \mathcal{O}(t^{-1}) \quad \text{as} \quad t \rightarrow \infty.$$

This bound had apparently never been published before. It says that the unperturbed chain behaves like a perfect insulator: the current integrated over arbitrarily long times remains bounded in $L^2(\mathbb{E}_\nu \mu_T)$.

2. The proof (see Section 3) shares some common features with a method used in [18] to obtain a weak coupling limit for noisy hamiltonian systems. In our case, one can indeed see the eigenmodes of the unperturbed system as weakly coupled by the noise and the anharmonic potentials.

3. The choice of fixed boundary conditions just turns out to be convenient for technical reasons (see Section 4).

4. The hypothesis that ν_- is large enough is only used to ensure the exponential decay of correlations of the Gibbs measure when $\lambda' > 0$.

Next, in the absence of anharmonicity ($\lambda' = 0$), one gets more refined results.

Theorem 2. *Let $\lambda > 0$, let $\lambda' = 0$, and let us assume that hypotheses introduced up to here hold. For almost all realizations of the pinnings, the Green-Kubo conductivity (2.5) of the chain is well defined, and in fact*

$$\kappa(\lambda, 0) = \frac{1}{T^2} \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X_{(\lambda, 0)}^s \, ds \right)^2, \quad (2.7)$$

this last limit being independent of the choice of boundary conditions (fixed or periodic). Moreover, there exists a constant $c > 0$ such that, for every $\lambda \in]0, 1[$,

$$c\lambda \leq \kappa(\lambda, 0) \leq c^{-1}\lambda. \quad (2.8)$$

This article is devoted to the proof of these theorems, which is constructed as follows.

Proof of Theorems 1 and 2. The upper bound (2.6) is derived in Section 3, assuming that Lemma 1 holds. This lemma is stated and shown in Section 4 ; it encapsulates the informations we need about the localization of the eigenmodes of the unperturbed system ($\lambda = \lambda' = 0$). The existence of $\kappa(\lambda, 0)$ for almost every realization of the pinnings, together with (2.7), are shown in Section 5. Finally, a lower bound on the conductivity when $\lambda' = 0$ is obtained in Section 6. This shows (2.8). \square

2.3 Heuristic comments

We would like to give here some intuition on the conductivity of disordered harmonic chains perturbed by a weak noise only, so with $\lambda > 0$ small and $\lambda' = 0$. We will develop in a more probabilistic way some ideas from [12]. Our results cover the case where the pinning parameters ν_k are bounded from below by a positive constant, but it could be obviously desirable to understand the unpinned chain as well, in which case randomness has to be puted on the value of the masses. We handle here both cases.

Let us first assume that $\nu_k \geq c$ for some $c > 0$, and let us consider a typical realization of the pinnings. In the absence of noise ($\lambda = 0$), the dynamics of the chain is actually equivalent to that of N independent one-dimensional harmonic oscillator, called eignmodes (see Subsection 4.1 and formulas (4.5-4.6) in particular). Since the chain is pinned at each site, the eigenfrequencies of these modes are uniformly bounded away from zero. As a result one expects all the modes to be exponentially localized. One can thus naively think that, to each particle, is associated a mode localized near the equilibrium position of this particle.

When the noise is turned on ($\lambda > 0$), energy starts being exchanged between near modes. Assume that, initially, energy is distributed uniformly between all the modes, except around the origin, where some more energy is added. We expect this extra amount of energy to diffuse with time, with a variance

proportional to $\kappa(0, \lambda) \cdot t$ at time t . Since flips of velocity occur at random times and with rate λ , one could compare the location of this extra energy at time t to the position of a standard random walk after $n = \lambda t$ steps. Therefore, denoting by δ_k the increments of this walk, one gets

$$\kappa(\lambda, 0) \sim \left\langle \left(\frac{1}{\sqrt{t}} \sum_{k=1}^n \delta_k \right)^2 \right\rangle \sim \lambda.$$

This intuitive picture will only be partially justified, as explained in the Remark after the proof of Theorem 1 in Section 3.

Let us now consider the unpinned chain. So we put $\nu_k = 0$ and we change p_k^2 by p_k^2/m_k in the Hamiltonian, where m_k are i.i.d. positive random variables. We consider a typical realization of the masses. In contrast with the pinned chain, the eigenfrequencies of the modes are now distributed in an interval of the form $[0, c]$, for some $c > 0$. This has an important consequence on the localization of the modes. One expects indeed that the localization length l of a mode and its eigenfrequency ω are related through the formula $l \sim 1/\omega^2$.

Here again, the noise induces exchange of energy between modes, and one still would like to compare $\kappa(\lambda, 0) \cdot t$ with the variance of a centred random walk with increments δ_k . However, due to the unlocalized low modes, δ_k can now take larger values than in the pinned case. Assuming that the eigenfrequencies are uniformly distributed in $[0, c]$, one guesses that, for large a ,

$$\mathbb{P}(|\delta_k| \geq a) \sim \mathbb{P}(1/\omega^2 \geq a) \sim 1/\sqrt{a}.$$

This however neglects a fact. Since energy does not travel faster than ballistically, and since successive flips of the velocity are spaced by time intervals of order $1/\lambda$, it is resonable to introduce the cutt-off $\mathbb{P}(|\delta_k| > 1/\lambda) = 0$. With this distribution for $|\delta_k|$, and with $n = \lambda t$, we now find

$$\kappa(\lambda, 0) \sim \left\langle \left(\frac{1}{\sqrt{t}} \sum_{k=1}^n \delta_k \right)^2 \right\rangle \sim \lambda^{-1/2}.$$

This scaling is numerically observed in [12]. The arguments leading to this conclusion are very approximative however, and it should be desirable to analyze this case rigorously as well.

3 Upper bound on the conductivity

We here proceed to the proof of Theorem 1. We assume that Lemma 1 in Section 4 holds: there exists a sequence $(u_N)_{N \geq 1} \subset L^2(\mathbb{E}_\nu \mu_T)$ such that $-A_{har} u_N = \mathcal{J}_{N, har}$, and that $(u_N)_{N \geq 1}$ and $(A_{anh} u_N)_{N \geq 1}$ are both bounded sequences in $L^2(\mathbb{E}_\nu \mu_T)$. Moreover u_N is of the form $u_N(q, p) = \langle q, \alpha_N q \rangle + \langle p, \gamma_N p \rangle + c_N$, where $\alpha, \gamma \in \mathbb{R}^N$ are symmetric matrices, and where $c_N \in \mathbb{R}$.

Proof of (2.6). Let $0 \leq \lambda' \leq \lambda$, and let u_N be the sequence obtained by Lemma 1 in Section 4. Before

starting, let us observe that, due to the special form of the function u_N , one has

$$A_{anh}u_N = \sum_{l=1}^N \phi_l(q) p_l, \quad (3.1)$$

with

$$\phi_l(q) = 2 \sum_{k=1}^N \gamma_{k,l} \cdot \left(\partial_x V(q_{k+1} - q_k) - \partial_x V(q_k - q_{k-1}) - \partial_x U(q_k) \right), \quad (3.2)$$

where $(\gamma_{k,l})_{1 \leq k, l \leq N}$ are the entries of γ_N . It follows in particular that

$$A_{anh}u_N = \frac{1}{2}(-S)A_{anh}u_N. \quad (3.3)$$

Now, since $\mathcal{J}_N = \mathcal{J}_{N,har} + \lambda' \mathcal{J}_{N,anh}$, one has, by Cauchy-Schwarz inequality,

$$\begin{aligned} \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X^s ds \right)^2 &\leq \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,har} \circ X^s ds \right)^2 + (\lambda')^2 \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,anh} \circ X^s ds \right)^2 \\ &\quad + 2\lambda' \left(\mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,har} \circ X^s ds \right)^2 \cdot \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,anh} \circ X^s ds \right)^2 \right)^{1/2}. \end{aligned}$$

Since $\mathcal{J}_{N,anh} = \frac{1}{2}(-S)\mathcal{J}_{N,anh}$, a classical bound [15] furnishes

$$\mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,anh} \circ X^s ds \right)^2 \leq C \mathbb{E}_\nu \mu_T(\mathcal{J}_{N,anh} \cdot (-\lambda S)^{-1} \mathcal{J}_{N,anh}) \leq \frac{C}{2\lambda} \mathbb{E}_\nu \mu_T(\mathcal{J}_{N,anh}^2)$$

where $C < +\infty$ is a universal constant. By (2.1), $\mathbb{E}_\nu \mu_T(\mathcal{J}_{N,anh}^2)$ is bounded uniformly in N . Therefore

$$\limsup_{t \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,anh} \circ X^s ds \right)^2 = \mathcal{O}(\lambda^{-1}).$$

It suffices thus to establish that

$$\limsup_{t \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{E}_\nu \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,har} \circ X^s ds \right)^2 = \mathcal{O}(\lambda).$$

One writes

$$\mathcal{J}_{N,har} = -A_{har}u_N = -Lu_N + \lambda' A_{anh}u_N + \lambda Su_N = -Lu_N + \lambda S \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N,$$

where the second equality is obtained by means of (3.3). Therefore

$$\begin{aligned} \frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_{N,har} \circ X^s ds &= \frac{-1}{\sqrt{t}} \int_0^t Lu_N \circ X^s ds + \frac{\lambda}{\sqrt{t}} \int_0^t S \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N \circ X^s ds \\ &= \frac{1}{\sqrt{t}} \mathcal{M}_t - \frac{u \circ X^t - u}{\sqrt{t}} + \frac{\lambda}{\sqrt{t}} \int_0^t S \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N \circ X^s ds, \end{aligned} \quad (3.4)$$

where \mathcal{M}_t is a martingale given by

$$\mathcal{M}_t = \int_0^t \sum_{j=1}^N S_j u_N \circ X_s (dN_s^j - \lambda ds),$$

with N_s^j the Poisson process that flips the momentum of particle j .

It now suffices to establish that the three terms in the right hand side of (3.4) are $\mathcal{O}(\lambda)$ in $L^2(\mathbf{E}_\nu \mathbf{E}_T)$. Up to the end of this proof, we use the simpler notation $\langle \cdot \rangle$, for $\mu_T(\cdot)$. Let us first show that $\langle u_N \cdot (-S)u_N \rangle \leq 4\|u_N\|_{L^2(\mu_T)}^2$. Writing

$$u_N = u_N^{p,p,0} + u_N^{p,p,1} + u_N^{q,q} + c_N$$

with

$$u_N^{p,p,0} = \sum_{i \neq j} \gamma_{i,j} p_i p_j, \quad u_N^{p,p,1} = \sum_i \gamma_{i,i} p_i^2, \quad u_N^{q,q} = \langle q, \alpha_N q \rangle,$$

one finds indeed

$$\langle u_N \cdot (-S)u_N \rangle = \langle u_N \cdot (-S)u_N^{p,p,0} \rangle = 4 \langle u_N^{p,p,0} \cdot u_N^{p,p,0} \rangle$$

and

$$\langle u_N \cdot u_N \rangle = \langle u_N^{p,p,0} \cdot u_N^{p,p,0} \rangle + \langle (u_N^{p,p,1} + u_N^{q,q} + c_N)^2 \rangle + 2 \langle (u_N^{p,p,1} + u_N^{q,q} + c_N) \cdot u_N^{p,p,0} \rangle.$$

The claim follows since $\langle (u_N^{p,p,1} + u_N^{q,q} + c_N) \cdot u_N^{p,p,0} \rangle = 0$.

So first,

$$\mathbb{E}_T \left(\frac{1}{\sqrt{t}} \mathcal{M}_t \right)^2 = 2\lambda \langle u_N, (-S)u_N \rangle \leq 8\lambda \|u_N\|_{L^2(\mu_T)}^2.$$

Next,

$$\mathbb{E}_T \left(\frac{u \circ X_t - u}{\sqrt{t}} \right)^2 \leq \frac{2}{t} \|u_N\|_{L^2(\mu_T)}^2.$$

Finally, by a classical bound [15],

$$\begin{aligned} \mathbb{E}_T \left(\frac{\lambda}{\sqrt{t}} \int_0^t S \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N \circ X_s \, ds \right)^2 &\leq C \lambda^2 \left\langle S \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N, (-\lambda S)^{-1} S \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N \right\rangle \\ &= C \lambda \left\langle \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N, (-S) \left(Id - \frac{\lambda'}{2\lambda} A_{anh} \right) u_N \right\rangle \\ &= C \lambda \left(\langle u_N, (-S)u_N \rangle + \frac{1}{2} \left(\frac{\lambda'}{\lambda} \right)^2 \|A_{anh} u_N\|_{L^2(\mu_T)}^2 \right) \\ &\leq C \lambda \left(\|u_N\|_{L^2(\mu_T)}^2 + \|A_{anh} u_N\|_{L^2(\mu_T)}^2 \right) \end{aligned}$$

where one has used (3.3) and

$$\langle u_N, A_{anh} u_N \rangle = \left\langle (\langle q, \alpha_N q \rangle + \langle p, \gamma_N p \rangle + c_N) \cdot \sum_{l=1}^N \phi_l(q) p_l \right\rangle = 0$$

to get the second equality. Taking the expectation over the pinnings, the proof is completed since $(u_N)_N$ and $(A_{anh} u_N)_N$ are bounded sequences in $L^2(\mathbf{E}_\nu \mu_T)$. \square

Remark. When $\lambda' = 0$, formula (3.4) becomes

$$\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X^s \, ds = \frac{1}{\sqrt{t}} \mathcal{M}_t - \frac{u_N \circ X^t - u_N}{\sqrt{t}} + \frac{\lambda}{\sqrt{t}} \int_0^t S u_N \circ X^s \, ds. \quad (3.5)$$

Now, since $Su_N = -4 \sum_{1 \leq k \neq l \leq N} \gamma_{k,l} p_k p_l$, one has

$$\int_0^t \mathcal{J}_N \circ X_{t-s} ds = - \int_0^t \mathcal{J}_N \circ X_s ds \quad \text{and} \quad \int_0^t Su_N \circ X_{t-s} ds = \int_0^t Su_N \circ X_s ds.$$

The measure on the paths being invariant under time reversal, one has

$$\mathbb{E}_T \left(\int_0^t \mathcal{J}_N \circ X_s ds \cdot \int_0^t Su_N \circ X_s ds \right) = 0.$$

Therefore, from (3.5), one deduces that

$$\mathbb{E}_T \left(\frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N(s) ds \right)^2 = \mathbb{E}_T \left(\frac{1}{\sqrt{t}} \mathcal{M}_t \right)^2 - \mathbb{E}_T \left(\frac{\lambda}{\sqrt{t}} \int_0^t Su_N \circ X_s ds \right)^2 + r(t)$$

where $r(t)$ is quantity that vanishes in the limit $t \rightarrow \infty$. We see thus that our proof does not completely justify the heuristic developed in Subsection 2.3, due to the second term in the right hand side of this last equation. As explained after the statment of Lemma 1 below, the sequence u_N should not be unique. It could be that a good choice of sequence u_N should make this second term to be $\mathcal{O}(\lambda^2)$.

4 Poisson equation for the unperturbed dynamics

In this section, we state and prove the folowing lemma. Fixed BC are assumed for the whole section.

Lemma 1. *Let $\lambda' \geq 0$, and assume fixed boundary conditions. For every $N \geq 1$, and for almost every realization of the pinnings, there exist a function u_N of the form*

$$u_N(q, p) = \langle q, \alpha_N q \rangle + \langle p, \gamma_N p \rangle + c_N,$$

where $\alpha_N, \gamma_N \in \mathbb{R}^{N \times N}$ are symmetric matrices and where $c_N \in \mathbb{R}$, such that

$$-A_{har} u_N = \mathcal{J}_{N,har}. \quad (4.1)$$

Moreover, the functions u_N can be taken so that

$$(u_N)_{N \geq 1} \quad \text{and} \quad (A_{anh} u_N)_{N \geq 1} \quad \text{are bounded sequences in } L^2(\mathbb{E}_\nu \mu_T).$$

Remarks. 1. The parameter λ' only plays a role through the definition of the measure μ_T .

2. For a given value of N and for almost every realization of the pinnings, the unperturbed dynamics is integrable, meaning here that it can be decomposed in N ergodic components, each of them corresponding to the motion of a single one dimensional harmonic oscillator (see Subsection 4.1 and (4.5-4.6) in particular). This has two implications. First, since (4.1) admits a solution, we conclude that the current \mathcal{J}_N is of mean zero with respect to the microcanonical measures of each ergodic component of the dynamics. Next, the solution u_N is not unique since every function f constant on the ergodic components of the dynamics satisfies $-A_{har} f = 0$.

Proof of Lemma 1. To simplify notations, we will generally not write the dependence on N explicitly. The proof is made of several steps.

4.1 Identifying $(u_N)_{N \geq 1}$: eigenmode expansion

Let $z > 0$ and let $1 \leq l, m \leq N$. Let us consider the equation

$$(z - A_{anh})u_{l,m,z} = q_l p_m.$$

The solution $u_{l,m,z}$ exists and is unique. It is given by

$$u_{l,m,z}(x) = \int_0^\infty e^{-zs} \cdot q_l \circ X_{(0,0)}^s(x) \cdot p_m \circ X_{(0,0)}^s(x) ds. \quad (4.2)$$

We will analyze $u_{l,m,z}$ to obtain the sequence u_N . Although we assumed fixed BC, all the results of this subsection apply for periodic BC as well.

Solutions to Hamilton's equations. The matrix Φ is a real symmetric positive definite matrix in $\mathbb{R}^{N \times N}$, and there exist thus an orthonormal basis $(\xi^k)_{1 \leq k \leq N}$ of \mathbb{R}^N , and a sequence of positive real numbers $(\omega_k^2)_{1 \leq k \leq N}$, such that

$$\Phi \xi^k = \omega_k^2 \xi^k.$$

One can show that

$$\min\{\nu_j : 1 \leq j \leq N\} \leq \omega_k^2 \leq \max\{\nu_j : 1 \leq j \leq N\} + 4 \quad (4.3)$$

for $1 \leq k \leq N$. According to Proposition II.1 in [16], for almost all realization of the pinnings, none of the eigenvalue is degenerate:

$$\omega_j \neq \omega_k \quad \text{if } j \neq k, \quad 1 \leq j, k \leq N. \quad (4.4)$$

In the sequel, we will assume that (4.4) holds.

When $\lambda = \lambda' = 0$, Hamilton's equations write

$$dq = p dt, \quad dp = -\Phi q dt.$$

For initial conditions (q, p) , the solutions write

$$q(t) = \sum_{k=1}^N \left(\langle q, \xi^k \rangle \cos \omega_k t + \frac{1}{\omega_k} \langle p, \xi^k \rangle \sin \omega_k t \right) \xi^k, \quad (4.5)$$

$$p(t) = \sum_{k=1}^N \left(-\omega_k \langle q, \xi^k \rangle \sin \omega_k t + \langle p, \xi^k \rangle \cos \omega_k t \right) \xi^k. \quad (4.6)$$

An expression for $u_{l,m,z}$. To determine $u_{l,m,z}$, one just needs to insert the solutions (4.5-4.6) in the definition (4.2), and then compute the integral, which is a sum of Laplace transforms of sines and

cosines:

$$\begin{aligned}
u_{l,m,z}(q,p) &= \frac{1}{4} \sum_{k=1}^N \langle l, \xi^k \rangle \langle m, \xi^k \rangle \left(-\langle q, \xi^k \rangle^2 + \frac{1}{\omega_k^2} \langle p, \xi^k \rangle^2 \right) \\
&+ \sum_{1 \leq j \neq k \leq N} \langle l, \xi^j \rangle \langle m, \xi^k \rangle \left(\frac{\omega_k^2}{\omega_j^2 - \omega_k^2} \langle q, \xi^j \rangle \langle q, \xi^k \rangle + \frac{1}{\omega_j^2 - \omega_k^2} \langle p, \xi^j \rangle \langle p, \xi^k \rangle \right) \\
&+ \mathcal{O}(z),
\end{aligned} \tag{4.7}$$

where $\langle j, \xi^k \rangle$ denotes the j^{th} component of the vector ξ^k , and where the rest term $\mathcal{O}(z)$ is a polynomial of the form $\langle q, \tilde{\alpha} q \rangle + \langle q, \tilde{\beta} p \rangle + \langle p, \tilde{\gamma} p \rangle$, where $\tilde{\alpha}$ and $\tilde{\gamma}$ can be taken to be symmetric. One defines

$$u_{l,m} = \lim_{z \rightarrow 0} u_{l,m,z}.$$

One observes that $u_{l,m}$ is of the form $\langle q, \tilde{\alpha} q \rangle + \langle p, \tilde{\gamma} p \rangle$ where $\tilde{\alpha}$ and $\tilde{\gamma}$ can be taken to be symmetric.

Defining the solution u_N . For fixed BC, the total current is given by

$$J_N = \frac{1}{2}(q_1 p_1 - q_N p_N) + \frac{1}{2} \sum_{k=1}^{N-1} (q_k p_{k+1} - q_{k+1} p_k)$$

Setting

$$w_l = u_{l,l-1} - u_{l-1,l} - \mu_T(u_{l,l-1} - u_{l-1,l})$$

for $2 \leq l \leq N$ and

$$w_1 = u_{N,N} - u_{1,1} - \mu_T(u_{N,N} - u_{1,1}),$$

we define

$$u_N = -\frac{1}{2\sqrt{N}} \sum_{k=1}^N w_k.$$

One observes that u_N is of the form $u_N = \langle q, \alpha_N q \rangle + \langle p, \gamma_N p \rangle + c_N$, where α_N and γ_N are symmetric matrices, and where $c_N \in \mathbb{R}$.

Let us show that u_N solves $-A_{anh} u_N = \mathcal{J}_N$. One may assume that $c_N = 0$ without loss of generality. The current \mathcal{J}_N can be written as $\mathcal{J}_N = \langle q, B p \rangle$. The function u_N has been obtained as the limit as $z \rightarrow 0$ of the function u_z of the form $u_z = \langle q, \alpha_z q \rangle + \langle q, \beta_z p \rangle + \langle p, \gamma_z p \rangle$ which solves $(z - A_{har}) u_z = \mathcal{J}_N$, and with α_z and γ_z symmetric. Since

$$(z - A_{har}) u_z = \langle q, (z \alpha_z + \beta_z \Phi) q \rangle + \langle q, (z \beta_z - 2(\alpha_z - \Phi \gamma_z)) p \rangle + \langle p, (z - \beta_z) p \rangle,$$

one has

$$z \alpha_z + \frac{1}{2}(\beta_z \Phi + \Phi \beta_z^\dagger) = 0, \quad z \beta_z - 2(\alpha_z - \Phi \gamma_z) = B, \quad z - \frac{1}{2}(\beta_z + \beta_z^\dagger) = 0.$$

One knows that $(\alpha_z, \beta_z, \gamma_z) \rightarrow (\alpha_N, 0, \gamma_N)$ as $z \rightarrow 0$, with α_N and γ_N symmetric, so that $-2(\alpha - \Phi \gamma) = B$. Taking into account that α_N and γ_N are symmetric, one gets

$$\Phi \gamma_N - \gamma_N \Phi = \frac{1}{2}(B - B^\dagger), \quad 2\alpha_N = 2\Phi \gamma_N - B. \tag{4.8}$$

It is checked that, if two symmetric matrices α_N and γ_N satisfy these relations, then $u_N = \langle q, \alpha_N q \rangle + \langle p, \gamma_N p \rangle$ solves the equation $-A_{har} u_N = \mathcal{J}_N$.

4.2 A new expression for w_l

For $1 \leq l \leq N$, let us write

$$w_l = \langle q, \alpha(l) q \rangle + \langle p, \gamma(l) p \rangle + c(l),$$

where $\alpha(l)$ and $\gamma(l)$ are symmetric matrices, and where $c(l) \in \mathbb{R}$. A relation similar to (4.8) is satisfied: defining the matrices

$$(B(l))_{m,n} = \delta_{l,l-1}(m,n) - \delta_{l-1,l}(m,n) \quad (2 \leq l \leq N) \quad \text{and} \quad (B(1))_{m,n} = \delta_{N,N}(m,n) - \delta_{1,1}(m,n),$$

for $1 \leq m, n \leq N$, one has

$$2\alpha(l) = 2\Phi\gamma(l) - B(l), \quad (4.9)$$

for $1 \leq l \leq N$. Therefore the knowledge of the matrices γ implies that of the matrices α .

An expression for the matrices $\gamma(l)$ can be recovered from (4.7) with $z = 0$. We will now work this out in order to obtain a more tractable formula. We show here that, for $2 \leq l \leq N$,

$$\begin{aligned} \gamma_{s,s}(l) &= - \sum_{j=l}^N \sum_{k=1}^N \langle s, \xi^k \rangle^2 \langle j, \xi^k \rangle^2, \quad 1 \leq s \leq l-1, \\ \gamma_{s,s}(l) &= \sum_{j=1}^{l-1} \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle^2, \quad l \leq s \leq N, \\ \gamma_{s,t}(l) &= \sum_{j=1}^{l-1} \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle \langle t, \xi^k \rangle, \quad 1 \leq s \neq t \leq N, \\ &= - \sum_{j=l}^N \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle \langle t, \xi^k \rangle, \quad 1 \leq s \neq t \leq N \end{aligned} \quad (4.10)$$

and

$$\gamma_{s,t}(1) = \frac{1}{4} \sum_{k=1}^N \frac{\langle N, \xi^k \rangle^2}{\omega_k^2} \langle s, \xi^k \rangle \langle t, \xi^k \rangle - \frac{1}{4} \sum_{k=1}^N \frac{\langle 1, \xi^k \rangle^2}{\omega_k^2} \langle s, \xi^k \rangle \langle t, \xi^k \rangle, \quad 1 \leq s, t \leq N. \quad (4.11)$$

Formula (4.11) is directly derived from (4.7), noting that $\gamma(1)$ is the only symmetric matrix such that $w_1(0, p) = \sum_{s,t} \gamma_{s,t}(1) p_s p_t$. To derive (4.10), one observes that $\gamma(l)$ is the only symmetric matrix such that $w_l(0, p) = \sum_{s,t} \gamma_{s,t}(l) p_s p_t$. Stating from (4.7), one deduces

$$w_l(0, p) = (u_{l,l-1} - u_{l-1,l})(0, p) = \sum_{1 \leq j \neq k \leq N} \left(\langle l, \xi^j \rangle \langle l-1, \xi^k \rangle - \langle l-1, \xi^j \rangle \langle l, \xi^k \rangle \right) \frac{\langle p, \xi^j \rangle \langle p, \xi^k \rangle}{\omega_j^2 - \omega_k^2}.$$

For fixed BC, the eigenvectors ξ^j satisfy the following relations for $1 \leq j \leq N$:

$$\begin{aligned} \langle \xi^j, 0 \rangle &= \langle \xi^j, N+1 \rangle = 0 \quad (\text{by definition}), \\ -\langle \xi^j, m-1 \rangle + (2 + \nu_m) \langle \xi^j, m \rangle - \langle \xi^j, m+1 \rangle &= \omega_j^2 \langle \xi^j, m \rangle, \quad 1 \leq m \leq N. \end{aligned}$$

So the following recurrence relation is satisfied:

$$\langle \xi^j, m+1 \rangle = (2 + \nu_m - \omega_j^2) \langle \xi^j, m \rangle - \langle \xi^j, m-1 \rangle, \quad 1 \leq m \leq N. \quad (4.12)$$

Let us first compute $w_2(0, p)$. Using (4.12), it comes

$$\begin{aligned} \langle 2, \xi^j \rangle \langle 1, \xi^k \rangle - \langle 1, \xi^j \rangle \langle 2, \xi^k \rangle &= (2 + \nu_1 - \omega_j^2) \langle 1, \xi^j \rangle \langle 1, \xi^k \rangle - (2 + \nu_1 - \omega_k^2) \langle 1, \xi^j \rangle \langle 1, \xi^k \rangle \\ &= -(\omega_j^2 - \omega_k^2) \langle 1, \xi^j \rangle \langle 1, \xi^k \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} w_2(0, p) &= - \sum_{1 \leq j \neq k \leq N} \langle 1, \xi^j \rangle \langle 1, \xi^k \rangle \langle p, \xi^j \rangle \langle p, \xi^k \rangle = - \sum_{1 \leq j, k \leq N} \langle 1, \xi^j \rangle \langle 1, \xi^k \rangle \langle p, \xi^j \rangle \langle p, \xi^k \rangle + \sum_{k=1}^N \langle 1, \xi^k \rangle^2 \langle p, \xi^k \rangle^2 \\ &= - \langle 1, p \rangle^2 + \sum_{k=1}^N \langle 1, \xi^k \rangle^2 \langle p, \xi^k \rangle^2, \end{aligned} \quad (4.13)$$

where the last equality follows from the fact that $(\xi^k)_k$ forms an orthonormal basis.

Let us now compute $w_l(0, p)$ for $2 < l \leq N$. Again by (4.12),

$$\begin{aligned} \langle l, \xi^j \rangle \langle l-1, \xi^k \rangle - \langle l-1, \xi^j \rangle \langle l, \xi^k \rangle &= \left((2 + \nu_{l-1} - \omega_j^2) \langle l-1, \xi^j \rangle - \langle l-2, \xi^j \rangle \right) \langle l-1, \xi^k \rangle \\ &\quad - \langle l-1, \xi^j \rangle \left((2 + \nu_{l-1} - \omega_k^2) \langle l-1, \xi^k \rangle - \langle l-2, \xi^k \rangle \right) \\ &= -(\omega_j^2 - \omega_k^2) \langle l-1, \xi^j \rangle \langle l-1, \xi^k \rangle \\ &\quad + \langle l-1, \xi^j \rangle \langle l-2, \xi^k \rangle - \langle l-2, \xi^j \rangle \langle l-1, \xi^k \rangle. \end{aligned}$$

Therefore

$$\begin{aligned} w_l(0, p) &= - \sum_{1 \leq j \neq k \leq N} \langle l-1, \xi^j \rangle \langle l-1, \xi^k \rangle \langle p, \xi^j \rangle \langle p, \xi^k \rangle + w_{l-1}(0, p) \\ &= - \langle l-1, p \rangle^2 + \sum_{k=1}^N \langle l-1, \xi^k \rangle^2 \langle p, \xi^k \rangle^2 + w_{l-1}(0, p). \end{aligned} \quad (4.14)$$

Combining (4.13) and (4.14), one arrives to an expression valid for $2 \leq l \leq N$:

$$w_l(0, p) = \sum_{j=1}^{l-1} \left(\sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle p, \xi^k \rangle^2 - \langle j, p \rangle^2 \right).$$

Let us now write $\langle j, p \rangle^2 = p_j^2$ and

$$\langle p, \xi^k \rangle^2 = \left(\sum_s p_s \langle s, \xi^k \rangle \right)^2 = \sum_{s, t} p_s p_t \langle s, \xi^k \rangle \langle t, \xi^k \rangle.$$

One gets

$$\begin{aligned}
w_l(0, p) &= \sum_{s,t} p_s p_t \sum_{j=1}^{l-1} \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle \langle t, \xi^k \rangle - \sum_{j=1}^{l-1} p_j^2 \\
&= \sum_{s=1}^{l-1} p_s^2 \left(\sum_{j=1}^{l-1} \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle^2 - 1 \right) \\
&\quad + \sum_{s=l}^N p_s^2 \sum_{j=1}^{l-1} \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle^2 \\
&\quad + \sum_{1 \leq s \neq t \leq N} p_s p_t \sum_{j=1}^{l-1} \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle \langle t, \xi^k \rangle.
\end{aligned}$$

In this formula, the coefficients of p_s^2 coincide with $\gamma_{s,s}(l)$ given by (4.10) for $l \leq s \leq N$, and the coefficients of $p_s p_t$ with $s \neq t$ coincide with the first expression of $\gamma_{s,t}(l)$ given by (4.10). To recover the coefficients $\gamma_{s,s}(l)$ for $1 \leq s \leq l-1$, just use the fact that $(\xi^k)_k$ and $(|k\rangle)_k$ are orthonormal basis:

$$\begin{aligned}
\sum_{j=1}^{l-1} \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle^2 - 1 &= \sum_{k=1}^N \left(1 - \sum_{j=l}^N \langle j, \xi^k \rangle^2 \right) \langle s, \xi^k \rangle^2 - 1 \\
&= \sum_{k=1}^N \langle s | \xi^k \rangle^2 - 1 - \sum_{j=l}^N \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle^2 = - \sum_{j=l}^N \sum_{k=1}^N \langle j, \xi^k \rangle^2 \langle s, \xi^k \rangle^2.
\end{aligned}$$

The second expression for the coefficients $\gamma_{s,t}(l)$ with $s \neq t$ in (4.10) is obtained by a similar trick.

4.3 Exponential bounds

We show here that there exist constants $C < +\infty$ and $c > 0$ independent of N such that

$$\mathbb{E}_\nu (\alpha_{j,k}^2(l)) \leq C \exp \left(-c(|j-l| + |k-l|) \right), \quad \mathbb{E}_\nu (\gamma_{j,k}^2(l)) \leq C \exp \left(-c(|j-l| + |k-l|) \right), \quad (4.15)$$

for $2 \leq l \leq N$ and for $1 \leq j, k \leq N$. This is still valid for $l = 1$ if one replaces $|k-l|$ by $\min\{|k-1|, |k-N|\}$ and $|j-l|$ by $\min\{|j-1|, |j-N|\}$. Due to (4.9), it suffices to establish these bounds for the matrices γ .

Let us first observe that the almost sure bounds

$$|\gamma_{s,t}(l)| \leq 1 \quad (2 \leq l \leq N), \quad |\gamma_{s,t}(1)| \leq \frac{1}{2 \min\{\omega_k^2 : 1 \leq k \leq N\}}$$

hold for $1 \leq s, t \leq N$. This is directly deduced from (4.10) and (4.11) by taking absolute values inside the sums if needed, using the bound $\sum_j \leq \sum_{j=1}^N$, using that $(\xi^k)_k$ and $(|k\rangle)_k$ are orthonormal basis, and Cauchy-Schwarz inequality if needed. By (4.3), $\min\{\omega_k^2 : 1 \leq k \leq N\} \geq c > 0$, where c does not depend on N . In particular $\mathbb{E}_\nu (|\gamma_{s,t}|^p) \leq C_p \mathbb{E}_\nu |\gamma_{s,t}|$ for every $p \geq 1$, so that we only need to bound $\mathbb{E}_\nu |\gamma_{s,t}|$.

We now will apply localization results originally derived by Kunz and Souillard [16], but we follow the exposition by [11]. From (4.10) and (4.11), we see that we are looking for upper bound on the absolute value of sums of the type

$$\sum_k \langle r, \xi^k \rangle^2 \langle t, \xi^k \rangle^2, \quad r < t,$$

and of the type

$$\sum_k \langle r, \xi^k \rangle^2 \langle s, \xi^k \rangle \langle t, \xi^k \rangle, \quad \sum_k \langle r, \xi^k \rangle \langle s, \xi^k \rangle^2 \langle t, \xi^k \rangle, \quad \sum_k \langle r, \xi^k \rangle \langle s, \xi^k \rangle \langle t, \xi^k \rangle^2, \quad r < s < t.$$

Since $|\langle r, \xi^k \rangle| \leq 1$ for $1 \leq r, k \leq N$, all of them can be bounded by

$$\sum_k |\langle r, \xi^k \rangle \langle t, \xi^k \rangle|.$$

By the formula before Lemma 4.3 in [11], and the lines after the proof of this lemma, one concludes that there exist constants $C < +\infty$ and $c > 0$ independent of N such that

$$\mathbb{E}_\nu \left(\sum_k |\langle r, \xi^k \rangle \langle t, \xi^k \rangle| \right) \leq C e^{-c(t-r)}.$$

Together with the remarks formulated up to here, this allows to deduce (4.15).

4.4 Concluding the proof of Lemma 1

One writes

$$\mathbb{E}_\nu \mu_T(u_N^2) = \frac{1}{4N} \sum_{m,n} \mathbb{E}_\nu \mu_T(w_m \cdot w_n) \quad \text{and} \quad \mathbb{E}_\nu \mu_T((A_{anh} u_N)^2) = \frac{1}{4N} \sum_{m,n} \mathbb{E}_\nu \mu_T(A_{anh} w_m \cdot A_{anh} w_n).$$

One will establish that there exist constants $C < +\infty$ and $c > 0$ such that

$$|\mathbb{E}_\nu \mu_T(w_m \cdot w_n)| \leq C e^{-c|m-n|} \quad \text{and} \quad |\mathbb{E}_\nu \mu_T(A_{anh} w_m \cdot A_{anh} w_n)| \leq C e^{-c|m-n|} \quad (4.16)$$

for $1 \leq m, n \leq N$. This will conclude the proof.

Let us fix $1 \leq m, n \leq N$. Let us first consider $|\mathbb{E}_\nu \mu_T(w_m \cdot w_n)|$. Let us observe that the functions w_l are of zero mean by construction, and so the relation

$$\sum_j \gamma_{j,j}(l) \int p_j^2 d\mu_T + \sum_{j,k} \alpha_{j,k}(l) \int q_j q_k d\mu_T + c(l) = 0 \quad (4.17)$$

holds for $1 \leq l \leq N$. Using this relation, one computes

$$\begin{aligned} \mu_T(w_m \cdot w_n) &= \left\langle \left(\sum_{i,j} \alpha_{i,j}(m) q_i q_j + \sum_{i,j} \gamma_{i,j}(m) p_i p_j + c(m) \right) \left(\sum_{i,j} \alpha_{i,j}(n) q_i q_j + \sum_{i,j} \gamma_{i,j}(n) p_i p_j + c(n) \right) \right\rangle_T \\ &= \sum_{i,j,k,l} \alpha_{i,j}(m) \alpha_{k,l}(n) \int q_i q_j q_k q_l d\mu_T + \sum_{i,j,k,l} \alpha_{i,j}(m) \gamma_{k,l}(n) \int q_i q_j p_k p_l d\mu_T \\ &\quad + \sum_{i,j,k,l} \gamma_{i,j}(m) \alpha_{k,l}(n) \int p_i p_j q_k q_l d\mu_T + \sum_{i,j,k,l} \gamma_{i,j}(m) \gamma_{k,l}(n) \int p_i p_j p_k p_l d\mu_T - c(m)c(n) \\ &= S_1 + S_2 + S_3 + S_4 - c(m)c(n). \end{aligned}$$

Using (4.17) and the fact that $\int p^2 d\mu_T = T$, the sum S_1 is rewritten as

$$\begin{aligned}
S_1 &= \sum_{i,j,k,l} \alpha_{i,j}(m) \alpha_{k,l}(n) \int \left(q_i q_j - \int q_i q_j d\mu_T \right) \left(q_k q_l - \int q_k q_l d\mu_T \right) d\mu_T \\
&\quad + \sum_{i,j,k,l} \alpha_{i,j}(m) \alpha_{k,l}(n) \int q_i q_j d\mu_T \int q_k q_l d\mu_T \\
&= \sum_{i,j,k,l} \alpha_{i,j}(m) \alpha_{k,l}(n) \int \left(q_i q_j - \int q_i q_j d\mu_T \right) \left(q_k q_l - \int q_k q_l d\mu_T \right) d\mu_T \\
&\quad + T^2 \sum_{i,j} \gamma_{i,i}(m) \gamma_{j,j}(n) + Tc(m) \sum_i \gamma_{i,i}(n) + Tc(n) \sum_i \gamma_{i,i}(m) + c(m)c(n).
\end{aligned}$$

Then, still using (4.17), one obtains

$$S_2 + S_3 = -Tc(m) \sum_i \gamma_{i,i}(n) - Tc(n) \sum_i \gamma_{i,i}(m) - 2T^2 \sum_{i,j} \gamma_{i,i}(m) \gamma_{j,j}(n).$$

Finally, the terms in the sum S_4 are non zero only when

$$i = j = k = l, \quad i = j, k = l, i \neq k, \quad i = k, j = l, i \neq j, \quad i = l, j = k, i \neq j.$$

Using that $\int p^4 d\mu_T = 3(\int p^2 d\mu_T)^2$ and that $\int p^2 d\mu_T = T$, S_4 is seen to be equal to

$$S_4 = T^2 \sum_{i,j} \left(\gamma_{i,i}(m) \gamma_{j,j}(n) + 2\gamma_{i,j}(m) \gamma_{i,j}(n) \right).$$

Therefore

$$\begin{aligned}
\mu_T(w_m \cdot w_n) &= \sum_{i,j,k,l} \alpha_{i,j}(m) \alpha_{k,l}(n) \int \left(q_i q_j - \int q_i q_j d\mu_T \right) \left(q_k q_l - \int q_k q_l d\mu_T \right) d\mu_T \\
&\quad + 2T^2 \sum_{i,j} \gamma_{i,j}(m) \gamma_{i,j}(n).
\end{aligned}$$

Applying the decorrelation bound (2.1) and the exponential estimate (4.15), one obtains the result.

Let us next consider $|\mathbb{E}_\nu \mu_T(A_{anh} w_m \cdot A_{anh} w_n)|$. From (3.1), one has

$$\mu_T(A_{anh} w_m \cdot A_{anh} w_n) = \sum_{s,t} \mu_T(\phi_t(q, m) \phi_s(q, n) p_s p_t) = T \sum_t \mu_T(\phi_t(q, m) \phi_t(q, n)).$$

Now, from (3.2), one has $\phi_t(q, k) = \sum_s \gamma_{s,t}(k) \rho_s(q)$ for $1 \leq k \leq N$. Here $\rho_s(q) = \rho_s(q_{s-1}, q_s, q_{s+1})$ is a function of mean zero since the potentials U and V are symmetric. One writes

$$\mu_T(A_{anh} w_m \cdot A_{anh} w_n) = \sum_t \sum_{s,s'} \gamma_{s,t}(m) \gamma_{s',t}(n) \mu_T(\rho_s \cdot \rho_{s'}).$$

Applying the decorrelation bound (2.1) and the exponential estimate (4.15), one obtains the result. \square

5 Convergence results

In this Section we show the convergence result (2.7). We assume thus $\lambda > 0$ and $\lambda' = 0$.

We start with some definitions (see [5] for details). The dynamics defined in Section 2 can also be defined for a set of particles indexed in \mathbb{Z} instead of \mathbb{Z}_N . Points on the phase space are written $x = (q, p)$, with $q = (q_k)_{k \in \mathbb{Z}}$ and $p = (p_k)_{k \in \mathbb{Z}}$. Let us denote by \mathcal{L} the generator of this infinite-dimensional dynamics, and by μ_T the corresponding Gibbs measure at temperature T . We extend the definition (2.2) of local currents j_k to all $k \in \mathbb{Z}$ ($j_k = j_{k,har}$ since $\lambda' = 0$). If $u = u(x, \nu)$, with $\nu = (\nu_k)_{k \in \mathbb{Z}}$ a sequence of pinnings, and if $k \in \mathbb{Z}$, we write $\tau_k u(x, \nu) = u(\tau_k x, \tau_k \nu)$, where

$$(\tau_k q)_j = q_{k+j}, \quad (\tau_k p)_j = p_{k+j}, \quad (\tau_k \nu)_j = \nu_{k+j}.$$

We will use the hermitian product $\langle u, v \rangle_N = \mu_T(u \cdot v^*)$, where v^* is the complex conjugate of v , and where $N \in \mathbb{N}$ or $N = \infty$ for the infinite dimensional dynamics. We will always write the index N to avoid confusion with the canonical scalar product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^d . We denote by $\ll \cdot, \cdot \gg$ the inner-product defined, for local bounded functions u and v , by

$$\ll u, v \gg = \sum_{k \in \mathbb{Z}} \mathbb{E}_\nu(\langle u, \tau_k v \rangle_\infty - \mu_T(u) \mu_T(v))$$

and by \mathcal{H} the corresponding Hilbert space, obtained by completion of the bounded local functions.

We start with two lemmas. We have no reason to think that Lemma 2 still holds if an anharmonic potential is added, and this is the main reason why we here restrict ourselves to harmonic interactions.

Lemma 2. *There exists a constant $C < +\infty$ such that, for any realization of the pinnings, for the finite dimensional dynamics with free or fixed B.C., or for the infinite dynamics, for any $k \geq 1$ and for any $l \in \mathbb{Z}_N$ (resp. $k \in \mathbb{Z}$ for the infinite dynamics), one has*

$$\|L^k j_l\|_{L^2(\mu_T)} \leq C^k \quad (\text{resp. } \|\mathcal{L}^k j_l\|_{L^2(\mu_T)} \leq C^k).$$

Proof. Let us consider the infinite dimensional dynamics ; other cases are similar. We can take $l = 0$ without loss of generality. The function j_0 is of the form $j_0 = \langle q, \alpha q \rangle + \langle q, \beta p \rangle + \langle p, \gamma p \rangle$, with $\alpha = \gamma = 0$ and β defined by

$$\beta_{i,j} = \frac{1}{2}(\delta_{0,0}(i,j) - \delta_{(1,1)}(i,j) + \delta_{0,1}(i,j) - \delta_{1,0}(i,j)).$$

Now, if u is any function of the type $u = \langle q, \alpha' q \rangle + \langle q, \beta' p \rangle + \langle p, \gamma' p \rangle$, then $\mathcal{L}u = \langle q, \alpha' q \rangle + \langle q, \beta' p \rangle + \langle p, \gamma' p \rangle$ with

$$(\alpha', \beta', \gamma') = \left(-\frac{\beta\Phi + \Phi\beta^\dagger}{2}, \alpha - 2\Phi\gamma - 2\lambda\beta, \frac{\beta + \beta^\dagger}{2} - 4\lambda\tilde{\gamma} \right),$$

where $\tilde{\gamma}$ is such that $(\tilde{\gamma})_{i,i} = 0$ and $(\tilde{\gamma})_{i,j} = \gamma_{i,j}$ for $i \neq j$. Thus

$$\mathcal{L}^k j_0 = \langle q, \alpha_{(k)} q \rangle + \langle q, \beta_{(k)} p \rangle + \langle p, \gamma_{(k)} p \rangle,$$

and there exists a constant $C < +\infty$ such that $\zeta_{i,j} = 0$ whenever $|i| \geq Ck$ or $|j| \geq Ck$ and such that $\zeta_{i,j} \leq C^k$ otherwise, with ζ one of the three matrices $\alpha_{(k)}$, $\beta_{(k)}$ or $\gamma_{(k)}$. The claim is obtained by expressing $\|\mathcal{L}^k j_l\|_{L^2(\mu_T)}$ in terms of the matrices $\alpha_{(k)}$, $\beta_{(k)}$ and $\gamma_{(k)}$. \square

Explicit representation for the matrix Φ^{-1} in Lemma 1.1. in [9] allows to deduce the following lemma.

Lemma 3. *Let f and g be two polynomials of the type $\langle q, \alpha q \rangle + \langle p, \beta p \rangle + \langle p, \gamma p \rangle$, and assume that there exists $n \in \mathbb{N}$ such that $\alpha_{i,j} = \beta_{i,j} = \gamma_{i,j} = 0$ whenever $|i| > n$ or $|j| > n$. Then there exists $c > 0$ such that, for fixed or periodic B.C.,*

$$|\langle f, g \rangle_N - \langle f, g \rangle_\infty| = \mathcal{O}(e^{-cN}) \quad \text{as } N \rightarrow \infty.$$

Let then $\mathbb{D} = \{z \in \mathbb{C} : \Re z > 0\}$. For every $z \in \mathbb{D}$, let u_z be the unique solution to the equation

$$(z - \mathcal{L})u_z = j_0. \quad (5.1)$$

We know from Theorem 1 in [5]¹, and from its proof, that

$$\lim_{z \rightarrow 0} \ll u_z, j_0 \gg \quad \text{exists and is finite} \quad (5.2)$$

and that

$$\lim_{z \rightarrow 0} z \ll u_z, u_z \gg = 0. \quad (5.3)$$

For $z \in \mathbb{D}$ and $N \geq 3$, let $u_{k,z,N}$ be the unique solution to the equation

$$(z - L)u_{k,z,N} = j_k. \quad (5.4)$$

so that

$$u_{z,N} := \frac{1}{\sqrt{N}} \sum_k u_{k,z,N} \quad \text{solves} \quad (z - L)u_{z,N} = \mathcal{J}_N. \quad (5.5)$$

Lemma 4. *For fixed or free boundary conditions and for almost all realizations of the pinnings,*

$$\begin{aligned} \lim_{z \rightarrow 0} \lim_{N \rightarrow \infty} \mu_T(u_{z,N} \cdot \mathcal{J}_N) &= \lim_{z \rightarrow 0} \lim_{N \rightarrow \infty} \mathbb{E}_\nu \mu_T(u_{z,N} \cdot \mathcal{J}_N) = \lim_{z \rightarrow 0} \ll u_z, j_0 \gg, \\ \lim_{z \rightarrow 0} \lim_{N \rightarrow \infty} z \mu_T(u_{z,N} \cdot u_{z,N}) &= \lim_{z \rightarrow 0} \lim_{N \rightarrow \infty} z \mathbb{E}_\nu \mu_T(u_{z,N} \cdot u_{z,N}) = 0. \end{aligned}$$

Proof. By (5.2) and (5.3), it suffices to establish separately that, for every $z \in \mathbb{D}$, and for almost every realization of the pinnings,

$$\begin{aligned} \lim_{N \rightarrow \infty} \mu_T(u_{z,N} \cdot \mathcal{J}_N) &= \ll u_z, j_0 \gg, & \lim_{N \rightarrow \infty} \mathbb{E}_\nu \mu_T(u_{z,N} \cdot \mathcal{J}_N) &= \ll u_z, j_0 \gg, \\ \lim_{N \rightarrow \infty} z \mu_T(u_{z,N} \cdot u_{z,N}) &= z \ll u_z, u_z \gg, & \lim_{N \rightarrow \infty} z \mathbb{E}_\nu \mu_T(u_{z,N} \cdot u_{z,N}) &= z \ll u_z, u_z \gg. \end{aligned}$$

The proof of these four relations is in fact very similar, and we will focus on the first one. We proceed in two steps: we first show the result for $|z|$ large enough, and then extend it to all $z \in \mathbb{D}$.

First step. Here we fix $z \in \mathbb{D}$ with $|z|$ large enough. We first assume periodic boundary conditions. The function u_z solving (5.1) may be given by

$$u_z = \sum_{k \geq 0} z^{-(k+1)} \mathcal{L}^k j_0,$$

¹ The model studied there is not exactly the same. The proof of the properties we mention here can be however readily adapted.

this series converging in virtue of Lemma 2 for $|z|$ large enough. Let now $n \geq 1$. One computes

$$\ll j_0, u_z \gg = \sum_{k=0}^n z^{-(k+1)} \sum_{l \in \mathbb{Z}} \mathbb{E}_\nu \langle j_l, L^k j_0 \rangle_\infty + \sum_{k=n+1}^{\infty} z^{-(k+1)} \sum_{l \in \mathbb{Z}} \mathbb{E}_\nu \langle j_l, L^k j_0 \rangle_\infty. \quad (5.6)$$

For every given k , the sum over l is actually a sum over Ck non-zero terms only, for some $C < +\infty$. From this fact and from Lemma 2, it is concluded that the second sum in the right hand side of (5.6) converges to 0 as $n \rightarrow \infty$. Similarly one writes

$$\langle \mathcal{J}_N, u_{z,N} \rangle_N = \frac{1}{N} \sum_{s,t} \sum_{k=0}^n z^{-(k+1)} \langle j_s, L^k j_t \rangle_N + \frac{1}{N} \sum_{s,t} \sum_{k=0}^n z^{-(k+1)} \langle j_s, L^k j_t \rangle_N. \quad (5.7)$$

Here as well, the second term in (5.7) is such that

$$\lim_{n \rightarrow \infty} \limsup_{N \rightarrow \infty} \frac{1}{N} \sum_{s,t} \sum_{k=0}^n z^{-(k+1)} \langle j_s, L^k j_t \rangle_N = 0.$$

To handle the first term in (5.7), let us write

$$F_n(\nu) = \sum_{t \in \mathbb{Z}_N} \sum_{k=0}^n z^{-(k+1)} \langle j_0, L^k j_t \rangle_N.$$

One has in fact

$$\frac{1}{N} \sum_{s,t} \sum_{k=0}^n z^{-(k+1)} \langle j_s, L^k j_t \rangle_N = \frac{1}{N} \sum_s F_n(\tau_s \nu).$$

The result is obtained by letting $N \rightarrow \infty$, invoking Lemma 3 and the ergodic theorem, and then letting $n \rightarrow \infty$. If one had started with fixed boundary conditions, then, for every fixed n , all the previous formulas remain valid up to some border terms that vanish in the limit $N \rightarrow \infty$ due to the factor $1/N$.

Second step. Denote by $\mathfrak{L}_{N,\nu}(z)$ and $\mathfrak{L}(z)$ the complex functions defined on \mathbb{D} by

$$\mathfrak{L}_{N,\nu}(z) = \langle u_{z,N}, \mathcal{J}_N \rangle_N \quad \text{and} \quad \mathfrak{L}(z) = \ll u_z, j_0 \gg.$$

The first observation is that these functions are well defined and analytic on \mathbb{D} . Moreover, similarly to what is proved in [5], they are uniformly bounded on \mathbb{D} by a constant independent of N and the realization of the pinning ν .

Let us fix a realization of the pinnings. The family $\{\mathfrak{L}_{N,\nu} ; N \geq 1\}$ is a normal family and by Montel's Theorem we can extract a subsequence $\{\mathfrak{L}_{N_k,\nu}\}_{k \geq 1}$ such that it converges (uniformly on every compact set of \mathbb{D}) to an analytic function f_ν^* .

By the first step we know that $f_\nu^*(z) = \mathfrak{L}(z)$ for any real $z > z_0$. Thus, since the functions involved are analytic, f_ν^* coincides with \mathfrak{L} on \mathbb{D} . It follows that the sequence $\{\mathfrak{L}_{N,\nu}(z)\}_{N \geq 1}$ converges for any $z \in \mathbb{D}$ to $\mathfrak{L}(z)$. \square

Following a classical argument, we can now proceed to the

Proof of (2.7). For any $z > 0$, one has

$$\begin{aligned} \frac{1}{\sqrt{t}} \int_0^t \mathcal{J}_N \circ X^s \, ds &= -\frac{1}{\sqrt{t}} \int_0^t L u_{z,N} \circ X^s \, ds + \frac{z}{\sqrt{t}} \int_0^t u_{z,N} \circ X^s \, ds \\ &= \frac{1}{\sqrt{t}} \mathcal{M}_{z,N,t} - \frac{u_{z,N} \circ X^t - u_{z,N}}{\sqrt{t}} + \frac{z}{\sqrt{t}} \int_0^t u_{z,N} \circ X^s \, ds \end{aligned}$$

Here $\mathcal{M}_{z,N,t}$ is a stationary martingale which variance is given by

$$\mathbb{E}_T(\mathcal{M}_{z,N,t}^2) = \mu_T(u_{z,N} \cdot (z - L)u_{z,N}) - z \mu_T(u_{z,N} \cdot u_{z,N}),$$

where one has used that $\mu_T(u_{z,N} \cdot A_{har} u_{z,N}) = 0$. Next one has

$$\mathbb{E}_T \left(\frac{u_{z,N} \circ X^t - u_{z,N}}{\sqrt{t}} \right)^2 \leq \frac{2}{t} \mu_T(u_{z,N} \cdot u_{z,N})$$

and

$$\mathbb{E}_T \left(\frac{z}{\sqrt{t}} \int_0^t u_{z,N} \circ X^s \, ds \right)^2 \leq z^2 t \mu_T(u_{z,N} \cdot u_{z,N}).$$

Reminding that $\mu_T(u_{z,N} \cdot (z - L)u_{z,N}) = \mu_T(u_{z,N} \cdot \mathcal{J}_N)$, the proof is completed by taking $z = 1/t$ and invoking Lemma 4. \square

6 Lower bound in the absence of anharmonicity

We here establish the lower bound in (2.8), and so we assume $\lambda > 0$ and $\lambda' = 0$. We also assume periodic boundary conditions. We use the same method as in [5] (see also [12]). According to Section 5, it is enough to establish that there exists a constant $c > 0$ such that, for almost every realization of the pinnings, for every $z > 0$ and for every $N \geq 3$, one has

$$\mu_T(\mathcal{J}_N \cdot (z - L)^2 \mathcal{J}_N) \geq c. \quad (6.1)$$

Proof of (6.1). For periodic B.C., the total current J_N is given by

$$J_N = \frac{1}{2} \sum_{k \in \mathbb{Z}_N} (q_k p_{k+1} - q_{k+1} p_k)$$

To get a lower bound on the conductivity, we use the following variational formula

$$\mu_T(J_N (z - L)^{-1} J_N) = \sup_f \{ 2\mu_T(J_N \cdot f) - \mu_T(f \cdot (z - \lambda S)f) - \mu_T(A_{har} f \cdot (z - \lambda S)^{-1} A_{har} f) \} \quad (6.2)$$

where the supremum is carried over the test functions $f \in \mathcal{C}_{temp}^\infty(\mathbb{R}^{2N})$. See [23] for a proof. We take f in the form

$$f = a \langle q, \beta p \rangle \quad \text{with} \quad a \in \mathbb{R} \quad \text{and} \quad \beta = \Phi M,$$

where M is the antisymmetric matrix such that $M_{i,j} = \delta_{i,j-1} - \delta_{i,j+1}$, with the convention of periodic B.C.: $\delta_{1,N+1} = \delta_{1,1}$ and $\delta_{N,0} = \delta_{N,N}$.

First, we have

$$A_{har}f = a\langle p, \beta p \rangle - a\langle q, \beta \Phi q \rangle = a \sum_{i \neq j} \beta_{i,j} p_i p_j$$

since $\beta \Phi = \Phi M \Phi$ is antisymmetric, and since $\beta_{i,i} = 0$ for $1 \leq i \leq N$. Since $S(p_i p_j) = -4p_i p_j$ for $i \neq j$, one obtains

$$\begin{aligned} \mu_T(Af \cdot (z - \lambda S)^{-1} Af) &= \frac{1}{z + 4\lambda} \mu_T(A_{har}f \cdot A_{har}f) = \frac{a^2 T^2}{z + 4\lambda} \sum_{i \neq j} (\beta_{i,j}^2 + \beta_{i,j} \beta_{j,i}) \\ &\leq C \frac{a^2 T^2 N}{z + 4\lambda} \end{aligned} \quad (6.3)$$

for some constant $C < +\infty$. Next, since $S p_k = -2p_k$ for $k \leq N$, there exists some constant $C < +\infty$ such that

$$\begin{aligned} \mu_T(f \cdot (z - \lambda S)f) &= a^2 T^2 (z + 2\lambda) \sum_{i,j} \beta_{i,j} (\Phi^{-1} \beta)_{i,j} = a^2 T^2 (z + 2\lambda) \text{Tr} [\beta^\dagger \Phi^{-1} \beta] \\ &\leq C a^2 T^2 (z + 2\lambda) N. \end{aligned} \quad (6.4)$$

Let us finally estimate the term $\mu_T(J_N f)$:

$$\begin{aligned} \mu_T(J_N f) &= \frac{a}{2} \left\langle \sum_{i,j} \beta_{i,j} q_i p_j \cdot \sum_{k \in \mathbb{Z}_N} (q_k p_{k+1} - q_{k+1} p_k) \right\rangle = \frac{a T^2}{2} \sum_{i,k} (\beta_{i,k+1} \langle q_i q_k \rangle - \beta_{i,k} \langle q_i q_{k+1} \rangle) \\ &= \frac{a T^2}{2} \sum_{k \in \mathbb{Z}_N} ((\beta^\dagger \Phi^{-1})_{k+1,k} - (\beta^\dagger \Phi^{-1})_{k,k+1}) = \frac{a T^2}{2} \sum_{k \in \mathbb{Z}_N} (M_{k,k+1} - M_{k+1,k}) \\ &= a T^2 N. \end{aligned} \quad (6.5)$$

By (6.3), (6.4), (6.5) and the variational formula (6.2), we find that there exists a constant $C < +\infty$, independent of the realization of the disorder, of λ and of N , such that for any positive a ,

$$\frac{1}{N T^2} \mu_T(J_N (z - L)^{-1} J_N) \geq a - C a^2 \left((z + 2\lambda) + \frac{1}{z + 4\lambda} \right).$$

By optimizing over a , this implies

$$\frac{1}{N T^2} \mu_T(J_N (z - L)^{-1} J_N) \geq \frac{1}{4C} \left((z + 2\lambda) + \frac{1}{z + 4\lambda} \right)^{-1}.$$

This concludes the proof. \square

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